Math 461: Topology, Fall 2009 — Final Exam, December 16

Name: .... *Solutions* ....

I	II	III	IV	V	VI	TOTAL
39	14	11	11	11	14	100

I. True or false? Please circle your answers.
(1.5 points for each correct answer, but **be careful**: 1 point will be subtracted for each wrong answer!)

1] *Math 461: Topology* is by far the best class you have ever taken. .... TRUE | FALSE

2] If $f\colon X \to Y$ is a function and X is countable, then $f(X)$ is countable. ...... **[TRUE]** FALSE

3] If $f\colon X \to Y$ is a function and Y is countable, then $f(X)$ is countable. .... **[TRUE]** FALSE

4] If $f\colon X \to Y$ is a function and Y is countable, then $f^{-1}(Y)$ is countable. .... TRUE **[FALSE]**

5] If X is a discrete topological space,
then any subspace of X is discrete. .... **[TRUE]** FALSE

6] If X is a topological space which is not discrete (i.e., not all subsets are open),
then **no** subspace of X is discrete. .... TRUE **[FALSE]**

7] If X is a disconnected topological space,
then any subspace of X is disconnected. .... TRUE **[FALSE]**

8] If X is a path-connected topological space,
then any subspace of X is path-connected. .... TRUE **[FALSE]**

9] If X is a compact topological space,
then any **closed** subspace of X is compact. .... **[TRUE]** FALSE

10] If X is a Hausdorff topological space,
then any subspace of X is Hausdorff. .... **[TRUE]** FALSE

11] If X is a second-countable topological space,
then any subspace of X is second-countable. .... **[TRUE]** FALSE

12] If X is a second-countable topological space,
then any **basis** for the topology of X is countable. .... TRUE **[FALSE]**

True or false? (Continued.)

13] If A and B are connected subspaces of a topological space X and $A \cap B \neq \emptyset$, then $A \cup B$ is connected. .... **[TRUE]** FALSE

14] If A and B are connected subspaces of a topological space X and $A \cap B \neq \emptyset$, then $A \cap B$ is connected. .... TRUE **[FALSE]**

15] If $f\colon X \to Y$ is a **continuous** function between topological spaces X and Y, then for every open subset U of X, $f(U)$ is open in Y. .... TRUE **[FALSE]**

16] If $f\colon X \to Y$ is a **homeomorphism** between topological spaces X and Y, then for every open subset U of X, $f(U)$ is open in Y. .... **[TRUE]** FALSE

17] If $f\colon X \to Y$ is a **bijective** function between topological spaces X and Y, and for every open subset U of X, $f(U)$ is open in Y, then f is a homeomorphism. TRUE **[FALSE]**

18] If X is a Hausdorff space, Y is a compact space, and $f\colon X \to Y$ is a continuous and bijective function, then f is a homeomorphism. .... TRUE **[FALSE]**

19] If X and Y are both **compact metric** spaces, and $f\colon X \to Y$ is a continuous and bijective function, then f is a homeomorphism. .... **[TRUE]** FALSE

20] $\mathbb{R}$ and $\mathbb{R}^2$ with the standard topologies are homeomorphic. .... TRUE **[FALSE]**

21] $\mathbb{Z}$ and $\mathbb{Z}^2$ with the discrete topologies are homeomorphic. .... **[TRUE]** FALSE

22] If $f\colon X \to Y$ is a continuous function between topological spaces X and Y, and X is connected and compact, then $f(X)$ is connected and compact. ..... **[TRUE]** FALSE

23] If $f\colon X \to Y$ is a continuous function between topological spaces X and Y, and X is separable, then $f(X)$ is separable. .... **[TRUE]** FALSE

24] If $f\colon X \to Y$ is a continuous function between topological spaces X and Y, and X is Hausdorff, then $f(X)$ is Hausdorff. .... TRUE **[FALSE]**

25] If $X = \mathbb{R}$ is given the cofinite (also known as finite complement) topology, then the function $f\colon X \to X$, $f(x) = \sin(x)$, is continuous. .... TRUE **[FALSE]**

26] If $X = \mathbb{R}$ is given the cofinite (also known as finite complement) topology, then the function $f\colon X \to X$, $f(x) = x^2$, is continuous. .... **[TRUE]** FALSE

[39 TOTAL POINTS]

II. Fill in the blanks in the following theorem.

Theorem. Let X and Y be topological spaces, and let $f\colon X \to Y$ be a function. Then the following conditions are equivalent:

(i) f is continuous, i.e., for every open subset U of Y, $f^{-1}(U)$ is ... *open in X* ....;

(ii) for every closed subset C of Y, $f^{-1}(C)$ is ... *closed in X* ....;

(iii) for every subset A of X, one has ... $f(\bar{A}) \subset \overline{f(A)}$ ....;

(iv) for every subset B of Y, one has ... $f^{-1}(\bar{B}) \supset \overline{f^{-1}(B)}$ ....;

(v) for every point $x \in X$ and every neighborhood V of $f(x)$ in Y, there is ....

.... *a neighborhood U of x in X such that $U \subset f^{-1}(V)$ (or equivalently $f(U) \subset V$).* ....

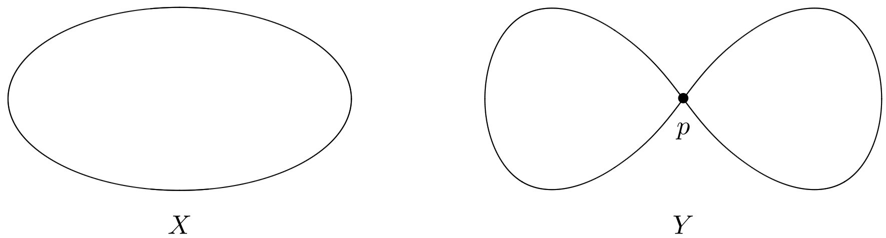
....

Prove **exactly two** implications of your choice from this theorem.

See the proof of theorem 18.1 in Munkres' book, pages 104–105.

14 TOTAL POINTS

III. Consider the following two subspaces of $\mathbb{R}^2$ with the standard topology.



Are X and Y homeomorphic? Justify your answer carefully.

The spaces X and Y are ***not*** *homeomorphic.*
Let p be the point of Y drawn in the picture above. Then $Y - \{p\}$ is disconnected, whereas for any point $q \in X$, $X - \{q\}$ is (path-)connected. So if there existed a homeomorphism $f : Y \to X$ then f would induce a homeomorphism between $Y - \{p\}$ and $X - \{f(p)\}$, which is impossible since $X - \{f(p)\}$ is connected but $Y - \{p\}$ is not.

11 TOTAL POINTS

IV. Complete the following definition.

Definition. If X is a topological space and A is a subset of X, then the *closure* of A in X is

$$\overline{A} = \left\{ x \in X \;\middle|\; \ldots\ldots\ldots\ldots\ldots\ldots \; \forall\, U \textit{ neighborhood of } x,\ A \cap U \neq \emptyset \; \ldots\ldots\ldots\ldots\ldots\ldots \right\}.$$

- If $X = \mathbb{R}^2$ with the **standard** topology and $A = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \text{ and } x \neq 0 \}$, then what is $\overline{A}$?

  $$\overline{A} = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \}.$$

- Now let X be a set and let A be a non-empty subset of X.

  The possible answers for the three following questions are as follows:

  (1) $\overline{A} = A$.

  (2) $\overline{A} = X$.

  (3) $\overline{A} = \begin{cases} A & \text{if } A \text{ is finite,} \\ X & \text{if } A \text{ is infinite.} \end{cases}$

  Write the number corresponding to the correct answer in each of the boxes below.

  - If X has the **indiscrete** topology and $\emptyset \neq A \subset X$, then 2
  - If X has the **discrete** topology and $\emptyset \neq A \subset X$, then 1
  - If X has the **cofinite** (also known as finite complement) topology and $\emptyset \neq A \subset X$, then 3

11 TOTAL POINTS

V. Complete the following two definitions, and then write the precise **statement (without proof!)** of **either** the intermediate value theorem **or** the extreme value theorem.

Definition. A topological space X is *disconnected* if and only if ...... $\exists\, U, V \subset X$ *such that* ......

.... U *and* V *are non-empty and open in* X, $U \cup V = X$, *and* $U \cap V = \emptyset$. ....

....

Definition. A topological space X is *compact* if and only if .... *every open cover of* X ....

...... *has a finite subcover, i.e.,* $\forall\, \mathscr{U} \subset \mathscr{P}(X)$, *if* $\forall\, U \in \mathscr{U}$, U *is open in* X *and* $\bigcup_{U \in \mathscr{U}} U = X$, ...

.... *then* $\exists\, n \in \mathbb{N}$ *and* $\exists\, U_1, U_2, \ldots, U_n \in \mathscr{U}$ *such that* $U_1 \cup U_2 \cup \cdots \cup U_n = X$. ....

....

Intermediate Value Theorem. Let X be a topological space, and let $f \colon X \to \mathbb{R}$ be a function.

Assume that X is ... *connected,* ....

and that f is ... *continuous.* ....

Then ... $\forall\, a, b \in X$ *and* $\forall\, y \in \mathbb{R}$, *if* $f(a) \leq y \leq f(b)$ *then* $\exists\, x \in X$ *such that* $f(x) = y$. ....

....

Extreme Value Theorem. Let X be a topological space, and let $f \colon X \to \mathbb{R}$ be a function.

Assume that X is ... *compact and not empty,* ....

and that f is ... *continuous.* ....

Then ... $\exists\, m, M \in X$ *such that* $\forall\, x \in X$, $f(m) \leq f(x) \leq f(M)$. ....

....

11 TOTAL POINTS

VI. Solve **only one** of the following two problems.

A] Recall that S^0 denotes the topological space with only two points $\{+1, -1\}$ and the discrete topology. Prove that a topological space X is disconnected if and only if there exists a continuous and surjective function $f\colon X \to S^0$.

Assume that X is disconnected. Then $\exists\, U, V \subset X$ such that U and V are non-empty and open in X, $U \cup V = X$, and $U \cap V = \emptyset$. Define $f\colon X \to S^0$, $f(x) = \begin{cases} -1 & \text{if } x \in U, \\ +1 & \text{if } x \in V. \end{cases}$
Since $U \cup V = X$ and $U \cap V = \emptyset$, f is well-defined. Since U and V are not empty, f is surjective. And since U and V are open, f is continuous.
Conversely, assume that $\exists\, f\colon X \to S^0$ continuous and surjective. Define $U = f^{-1}(\{-1\})$ and $V = f^{-1}(\{+1\})$. Since f is continuous, U and V are open in X. Since f is surjective, U and V are not empty. And finally we have

$$U \cup V = f^{-1}(\{-1\}) \cup f^{-1}(\{+1\}) = f^{-1}(\{-1\} \cup \{+1\}) = f^{-1}(S^0) = X, \text{ and}$$
$$U \cap V = f^{-1}(\{-1\}) \cap f^{-1}(\{+1\}) = f^{-1}(\{-1\} \cap \{+1\}) = f^{-1}(\emptyset) = \emptyset.$$

B] Recall the following result that we proved in class.

Lemma. If C is a compact subset of a Hausdorff space X and $x \in X - C$, then there exist open subsets U and V of X such that $C \subset U$, $x \in V$, and $U \cap V = \emptyset$.

Now let X be a Hausdorff space, and let C and D be compact subsets of X such that $C \cap D = \emptyset$. Prove that there exist open subsets U and V of X such that $C \subset U$, $D \subset V$, and $U \cap V = \emptyset$.

The lemma implies that $\forall\, x \in D$, $\exists\, U_x, V_x$ open in X such that $C \subset U_x$, $x \in V_x$, and $U_x \cap V_x = \emptyset$. Then $\{V_x\}_{x \in D}$ is an open cover of D, and therefore, since D is compact, $\exists\, n \in \mathbb{N}$, $\exists\, x_1, \ldots, x_n \in D$ such that $V_{x_1} \cup \cdots \cup V_{x_n} \supset D$. Define $V = V_{x_1} \cup \cdots \cup V_{x_n}$ and $U = U_{x_1} \cap \cdots \cap U_{x_n}$. Then $D \subset V$ and V is open in X since it is a union of open sets; $C \subset U$ since $\forall\, 1 \leq i \leq n$, $C \subset U_{x_i}$, and U is open in X since it is a finite intersection of open sets; and finally $U \cap V = \emptyset$ because if $y \in V$ then $\exists\, 1 \leq i \leq n$ such that $y \in V_{x_i}$, hence $y \notin U_{x_i}$ since $U_{x_i} \cap V_{x_i} = \emptyset$, and so $y \notin U$.

Have a great winter break!

14 TOTAL POINTS