Math 330: Intro to Higher Math, Section 1, Spring 2009 — Project Assignment, due May 4

On the set $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ define a relation $\sim$ by declaring $(a, b) \sim (c, d)$ if and only if $ad = bc$.

Problem 1. *Prove that* $\sim$ *is an equivalence relation.*

For all $(a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$ define $(a, b) + (c, d) = (ad + bc, bd)$ and $(a, b) \cdot (c, d) = (ac, bd)$.

Problem 2. *Prove that if* $(a_1, b_1) \sim (a_2, b_2)$ *and* $(c_1, d_1) \sim (c_2, d_2)$*, then* $(a_1, b_1)+(c_1, d_1) \sim (a_2, b_2)+(c_2, d_2)$ *and* $(a_1, b_1) \cdot (c_1, d_1) \sim (a_2, b_2) \cdot (c_2, d_2)$.

Let $[a, b]$ denote the equivalence class with respect to $\sim$ of $(a, b) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$, and define $\mathbf{Q}$ to be the set of equivalence classes of $\mathbb{Z} \times (\mathbb{Z} - \{0\})$.

For all $[a, b], [c, d] \in \mathbf{Q}$ define $[a, b] + [c, d] = [(a, b) + (c, d)]$ and $[a, b] \cdot [c, d] = [(a, b) \cdot (c, d)]$; these definitions make sense, i.e., they do not depend on the choice of representatives, because of problem 2.

The following properties hold for all $[a, b], [c, d], [e, f] \in \mathbf{Q}$:

(i) $([a, b] + [c, d]) + [e, f] = [a, b] + ([c, d] + [e, f])$;
(ii) $[a, b] + [c, d] = [c, d] + [a, b]$;
(iii) $([a, b] \cdot [c, d]) \cdot [e, f] = [a, b] \cdot ([c, d] \cdot [e, f])$;
(iv) $[a, b] \cdot [c, d] = [c, d] \cdot [a, b]$;
(v) $([a, b] + [c, d]) \cdot [e, f] = ([a, b] \cdot [e, f]) + ([c, d] \cdot [e, f])$;
(vi) there exists $\mathbf{0} \in \mathbf{Q}$ such that for all $[a, b] \in \mathbf{Q}$, $\mathbf{0} + [a, b] = [a, b]$;
(vii) for every $[a, b] \in \mathbf{Q}$ there exists $-[a, b] \in \mathbf{Q}$ such that $-[a, b] + [a, b] = \mathbf{0}$;
(viii) there exists $\mathbf{1} \in \mathbf{Q} - \{\mathbf{0}\}$ such that for all $[a, b] \in \mathbf{Q}$, $\mathbf{1} \cdot [a, b] = [a, b]$;
(ix) for every $[a, b] \in \mathbf{Q} - \{\mathbf{0}\}$ there exists $[a, b]^{-1} \in \mathbf{Q}$ such that $[a, b]^{-1} \cdot [a, b] = \mathbf{1}$.

Problem 3. *Prove properties* (vi), (vii), (viii), *and* (ix) *(notice that in particular you need to explicitly define* $\mathbf{0}, -[a, b], \mathbf{1}, [a, b]^{-1}$*). Prove also at least one of the remaining properties.*

Problem 4. *Among the properties* (i)–(ix) *above, which ones hold and which ones fail in* $\mathbb{Z} \times (\mathbb{Z} - \{0\})$*? Justify your answer.*

For $[a, b], [c, d] \in \mathbf{Q}$ define $[a, b] \leq [c, d]$ if and only if ($bd > 0$ and $ad \leq bc$) or ($bd < 0$ and $ad \geq cb$).

Problem 5. *Prove that the above definition of* $\leq$ *does not depend on the choice of representatives.*

Problem 6. *Prove that the following properties hold for all* $[a, b], [c, d], [e, f] \in \mathbf{Q}$*:*
(a) $[a, b] \leq [c, d]$ *or* $[c, d] \leq [a, b]$*;*
(b) if $[a, b] \leq [c, d]$ *and* $[c, d] \leq [a, b]$*, then* $[a, b] = [c, d]$*;*
(c) if $[a, b] \leq [c, d]$ *and* $[c, d] \leq [e, f]$*, then* $[a, b] \leq [e, f]$.

Problem 7. *Define a function* $f \colon \mathbb{Z} \to \mathbf{Q}$ *satisfying the following properties:*
(A) $f(0) = \mathbf{0}$ *and* $f(1) = \mathbf{1}$*;*
(B) for all $m, n \in \mathbb{Z}$, $f(m + n) = f(m) + f(n)$, $f(m \cdot n) = f(m) \cdot f(n)$*, and if* $m \leq n$ *then* $f(m) \leq f(n)$*;*
(C) for all $[a, b] \in \mathbf{Q}$ *there exists* $n \in \mathbb{N}$ *such that* $[a, b] \leq f(n)$.
Prove also that f *is injective but not surjective.*