Math 330: Intro to Higher Math, Section 1, Spring 2009 — Homework # 5, due February 9

Name: ....

In this homework assignment you will explore the possibility of defining addition and multiplication operations for "things" that are *not* the integers, in such a way that the axioms of chapter 1 are still satisfied.

We want to use only four "things", i.e., we want to use a set with precisely four elements. The axioms of chapter 1 tell us that we need a 0 and a 1, and that $0 \neq 1$. Let's call the remaining two elements ∇ and $\bigstar$. So from now on we will be working with the set with exactly four elements: $0, 1, \nabla, \bigstar$.

In order to describe the addition and multiplication we will produce operation tables.

First attempt. Let's try to define the addition as follows.

$+$	0	1	∇	$\bigstar$
0	0	1	∇	$\bigstar$
1	1	∇	$\bigstar$	0
∇	∇	$\bigstar$	0	1
$\bigstar$	$\bigstar$	0	1	∇

Notice that if we want axiom 1.2 (about the identity element for addition) to hold, then we have no choice for the first row and for the first column. Moreover, axiom 1.1(i) (the commutativity property of addition) tells us that the table must be symmetric with respect to the diagonal from the upper left to the bottom right corner. You should think about what other restrictions are imposed by the remaining axioms.

Now you have to fill in the two tables below. From the addition table you can read what the additive inverse of any given element is. For the multiplication table, you should argue as above and convince yourself that only the 2-by-2 square in the bottom right corner needs to be determined; but axiom 1.1(iii) (the distributive property) leaves you no choice. Explain how you determine $\nabla \cdot \nabla$, $\nabla \cdot \bigstar$, and $\bigstar \cdot \bigstar$.

x	$-x$
0	
1	
∇	
$\bigstar$	

$\cdot$	0	1	∇	$\bigstar$
0				
1				
∇				
$\bigstar$				

$\nabla \cdot \nabla =$ ....

$\nabla \cdot \bigstar =$ ....

$\bigstar \cdot \bigstar =$ ....

Unfortunately one of the axioms of chapter 1 is *not* fulfilled: Which one? ....
Why? ....

Second attempt. It turns out that it is in fact possible to define addition and multiplication operations on the set with four elements 0, 1, ∇, $\bigstar$, in such a way that *all* the axioms of chapter 1 are satisfied. Complete the following tables, and explain how you determine $\nabla + \nabla$.

$+$	0	1	∇	$\bigstar$
0				
1				
∇				
$\bigstar$				

x	$-x$
0	
1	1
∇	
$\bigstar$	

$\cdot$	0	1	∇	$\bigstar$
0				
1				
∇			$\bigstar$	
$\bigstar$				

$\nabla + \nabla =$ ....