Math 330: Introduction to Higher Math, Section 1, Spring 2009 — Final Exam, May 11

Name: ....

I	II	III	IV	V	VI	VII	TOTAL
11	10	14	23	14	14	14	100

I. Let X, Y, and Z be statements.
Are the following statements equivalent to "If X is true, then Y is true or Z is true"?
Please circle your answers.

1] If X is true and Y is false, then Z is true. .... YES | NO

2] If X is true and Z is false, then Y is true. .... YES | NO

3] If Y is false and Z is false, then X is false. .... YES | NO

4] If Y is false or Z is false, then X is false. .... YES | NO

5] If Y is true or Z is true, then X is true. .... YES | NO

6] If X is true, then Y is true and Z is true. .... YES | NO

7] If X is false, then Y is false and Z is false. .... YES | NO

8] If X is false, then Y is false or Z is false. .... YES | NO

9] X is false or Y is true or Z is true. .... YES | NO

10] X is true and Y is true and Z is true. .... YES | NO

11] X is true and Y is true, or X is true and Z is true. .... YES | NO

II. For each of the following five questions, four possible answers are provided, but only one of them is correct: write the corresponding letter in the box!

II/1. Let $f\colon X \to Y$ be a function. Let x and x' be elements of X such that $f(x) = f(x')$.
What do we need to know about f to conclude that $x = x'$? .... ☐

A] Nothing: this is true for all functions f.
B] We need f to be injective.
C] We need f to be surjective.
D] We need f to be bijective.

II/2. Let $f\colon X \to Y$ be a function. Let x and x' be elements of X such that $x = x'$.
What do we need to know about f to conclude that $f(x) = f(x')$? .... ☐

A] Nothing: this is true for all functions f.
B] We need f to be injective.
C] We need f to be surjective.
D] We need f to be bijective.

II/3. Let $f\colon X \to Y$ be a function. Let y be an element of Y.
What do we need to know about f to conclude that $y = f(x)$ for some $x \in X$? .... ☐

A] Nothing: this is true for all functions f.
B] We need f to be injective.
C] We need f to be surjective.
D] We need f to be bijective.

II/4. Let $f\colon X \to Y$ be a function. Let y be an element of Y.
What do we need to know about f to conclude that $y = f(x)$ for exactly one $x \in X$? ..... ☐

A] Nothing: this is true for all functions f.
B] We need f to be injective.
C] We need f to be surjective.
D] We need f to be bijective.

II/5. Let $f\colon X \to Y$ be a function. Let y be an element of Y.
What do we need to know about f to conclude that $y = f(x)$ for at most one $x \in X$? ..... ☐

A] Nothing: this is true for all functions f.
B] We need f to be injective.
C] We need f to be surjective.
D] We need f to be bijective.

III. Let a and b be integers, i.e., $a, b \in \mathbb{Z}$.

III/1. What exactly does it mean to say that "a is divisible by b", or equivalently that "b divides a"?

III/2. Is it true or false that for every natural number $n \in \mathbb{N}$, 6^n is not divisible by 5? ..... [ ]
Prove your claim.

IV. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

IV/1. What exactly does it mean to say that $(x_n)_{n\in\mathbb{N}}$ is convergent?

IV/2. Prove the following statement: If $(x_n)_{n\in\mathbb{N}}$ is convergent, then for every $\varepsilon \in \mathbb{R}_{>0}$ there exists an $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, if $m \geq N$ and $n \geq N$ then $|x_m - x_n| < \varepsilon$.

IV/3. What is the contrapositive of the statement in IV/2?

IV/4. Use IV/3 to show that the sequence $x_n = (-1)^n$ is divergent.

V. For each natural number $n \in \mathbb{N}$, define $x_n = \sum_{j=1}^{n} \frac{1}{j^2}$.

V/1. Prove that for all $n \in \mathbb{N}$, $x_n \leq 2 - \frac{1}{n}$.

V/2. Does the sequence $(x_n)_{n \in \mathbb{N}}$ defined above converge in $\mathbb{R}$? .... ☐

Prove your claim.

VI. VI/1. Define a relation $\sim$ on the set of real numbers $\mathbb{R}$ as follows: for all $x, y \in \mathbb{R}$, declare $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Is $\sim$ an equivalence relation? ....

Prove your claim.

VI/2. More generally, let A be a subset of $\mathbb{R}$ and define a relation $\backsim$ on $\mathbb{R}$ by declaring $x \backsim y$ if and only if $x - y \in A$. What conditions must A satisfy in order for $\backsim$ to be an equivalence relation?

VII. Suppose that you have a set A and a subset $B \subseteq A$ such that $B \neq A$.

VII/1. What exactly do these conditions mean?

$B \subseteq A$

$B \neq A$

VII/2. Given A and B satisfying the above conditions, is it possible for A and B to have the same cardinality, i.e., $A \simeq B$? .... ☐

A] No, it is not possible for any A.
B] Yes, it is possible for any A.
C] Yes, but only if A is empty.
D] Yes, but only if A is not empty.
E] Yes, but only if A is finite.
F] Yes, but only if A is infinite.
G] Yes, but only if A is countable.
H] Yes, but only if A is uncountable.

VII/3. Prove that your answer to VII/2 is correct.