Math 330: Introduction to Higher Math, Section 1, Spring 2009 – Midterm Exam, March 12

Name: ..........................................................................................

| I | II | III | IV/1 | IV/2 | V | VI/1 | VI/2 | TOTAL |
|---|---|---|---|---|---|---|---|---|
| 10 | 10 | 12 | 4 | 20 | 20 | 4 | 20 | 100 |
| | | | | | | | | |

I. For each of the following three questions, seven possible answers are provided, but only one of them is correct: write the corresponding letter in the box!
The symbols $X$, $Y$, and $Z$ represent here arbitrary statements.

I/1. If you know that $X$ implies $Y$, then you can also conclude that: ..........................

A] $X$ is true, and $Y$ is also true.
B] $X$ cannot be false.
C] $Y$ cannot be false.
D] At least one of $X$ and $Y$ is true.
E] If $Y$ is true, then $X$ is true.
F] If $Y$ is false, then $X$ is false.
G] If $X$ is false, then $Y$ is false.

I/2. Which of the following strategies is *not* a valid way to show that "$X$ implies $Y$"? ..........

A] Assume that $X$ is true, and then use this to show that $Y$ is true.
B] Assume that $Y$ is false, and then use this to show that $X$ is false.
C] Show that either $X$ is false, or $Y$ is true, or both.
D] Assume that $X$ is true and $Y$ is false, and deduce a contradiction.
E] Assume that $X$ is false and $Y$ is true, and deduce a contradiction.
F] Show that $X$ implies some intermediate statement $Z$, and then show that $Z$ implies $Y$.
G] Show that some intermediate statement $Z$ implies $Y$, and then show that $X$ implies $Z$.

I/3. If you want to *disprove* the claim that "$X$ implies $Y$", you need to show* that: ............

A] $Y$ is true, but $X$ is false.
B] $X$ is true, but $Y$ is false.
C] $X$ is false.
D] $Y$ is false.
E] Both $X$ and $Y$ are false.
F] Exactly one of $X$ and $Y$ is false.
G] At least one of $X$ and $Y$ is false.

*Beware of the difference between "you need to show ..." and "in certain cases, but not in general, it would be enough to show ..."!

Problem I is taken from a quiz by Terence Tao at http://scherk.pbwiki.com/

II. Consider the statement $X =$ "If I am taking Math 330, then I love math or I am a masochist".

II/1. What is the contrapositive of $X$?

II/2. What is the negation of $X$?

II/3. Is $X$ logically equivalent to the statement "If I am taking Math 330 and I do not love math, then I am a masochist"? Answer YES or NO. ........................................ $\boxed{\phantom{XXXX}}$

III. What are the negations of the following statements?

III/1. Math 330 is fun and not hard.

III/2. For all real numbers $x$ and $y$, if $x < y$ then there exists a rational number $q$ such that $x < q < y$.

III/3. $n$ is even if and only if $n^2$ is even.

IV. Let $k$ and $n$ be integers, i.e., $k, n \in \mathbb{Z}$.

IV/1. What exactly does it mean to say that "$k$ is divisible by $n$", or equivalently that "$n$ divides $k$"?

IV/2. Is -2 divisible by 3? Carefully justify your answer.

V. Recall that an integer $n \in \mathbb{Z}$ is called *even* if it is divisible by 2, and it is called *odd* if it is not even. Recall also that we already proved that $n$ is odd if and only if there exists an even integer $e$ such that $n = e + 1$.

Prove that for all integers $m$ and $n$, $mn$ is even if and only if at least one of $m$ and $n$ is even.

VI. A sequence $a_1$, $a_2$, $a_3$, $\ldots$ of integers is defined recursively as follows:

- $a_1 = 0$;
- For each $n \in \mathbb{N}$, $a_{n+1} = 2a_n + n$.

VI/1. Compute $a_2$, $a_3$, $a_4$, and $a_5$.

| $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|
| | | | |

VI/2. Prove that for all $n \in \mathbb{N}$, $a_n = 2^n - n - 1$.