Math 222 Calculus 2 Fall 2008 Exam 2 Solutions

(1) (40 Points, 8 Points Each) Evaluate the following integrals.

(a) $\int x\cos(x)\ dx$

Using integration by parts with $u = x$ and $dv = \cos(x)dx$, we have $du = dx$ and $v = \sin(x)$, so

$$\int x\cos(x)\ dx = x\sin(x) - \int \sin(x)dx = x\sin(x) + \cos(x) + C$$

(b) $\int \tan^8(x)\ \sec^4(x)\ dx$

Let $u = \tan(x)$ so $du = \sec^2(x)dx$, and $\sec^2(x) = 1 + \tan^2(x) = 1 + u^2$, so the integral becomes

$$\int \tan^8(x)\ \sec^4(x)\ dx = \int \tan^8(x)\ \sec^2(x)\ \sec^2(x)\ dx = \int u^8(1+u^2)\ du$$

$$= \int (u^8 + u^{10})\ du = \frac{u^9}{9} + \frac{u^{11}}{11} + C = \frac{\tan^9(x)}{9} + \frac{\tan^{11}(x)}{11} + C$$

(c) $\int e^{2x}\cos(x)dx$

Use integration by parts twice, the first time with $u = e^{2x}$ and $dv = \cos(x)dx$, so that $du = 2e^{2x}$ and $v = \sin(x)$, giving

$$\int e^{2x}\cos(x)dx = e^{2x}\sin(x) - 2\int e^{2x}\sin(x)\ dx$$

For the second integration by parts on the remaining integral, use $u = e^{2x}$ and $dv = \sin(x)\ dx$, so that $du = 2e^{2x}$ and $v = -\cos(x)$, giving

$$\int e^{2x}\cos(x)dx = e^{2x}\sin(x) - 2\left[-e^{2x}\cos(x) - \int(-\cos(x)2e^{2x})dx\right]$$

$$= e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4\int e^{2x}\cos(x)dx$$

Bringing the last term to the other side of the equation, and then dividing by 5 gives the answer

$$\int e^{2x}\cos(x)dx = \frac{1}{5}\left[e^{2x}\sin(x) + 2e^{2x}\cos(x)\right] + C$$

(d) $\displaystyle\int \frac{3x^2-x+1}{x(x^2+1)}\,dx = \int\left(\frac{A}{x}+\frac{Bx+C}{x^2+1}\right)dx$ (partial fractions) where

$$3x^2-x+1=(A)(x^2+1)+(Bx+C)x=(A+B)x^2+Cx+A$$

gives the equations $A+B=3$, $C=-1$, $A=1$ so $A=1$, $B=2$, $C=-1$. Then the integral is

$$\begin{aligned}\int \frac{3x^2-x+1}{x(x^2+1)}\,dx &= \int \frac{1}{x}\,dx+\int\frac{2x}{x^2+1}\,dx-\int\frac{1}{x^2+1}\,dx\\ &=\ln|x|+\ln|x^2+1|-\tan^{-1}(x)+C\\ &=\ln|x(x^2+1)|-\tan^{-1}(x)+C \quad \text{either expression is correct.}\end{aligned}$$

(e) $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2\,\sqrt{4-x^2}}\,dx$

Since $\cos^2(u)=1-\sin^2(u)$, use the trig substitution $x=2\sin(u)$ so $\sqrt{4-x^2}=\sqrt{4-4\sin^2(u)}=2\cos(u)$ and $dx=2\cos(u)du$. The limits of integration also change: $x=1\leftrightarrow u=\frac{\pi}{6}$ and $x=\sqrt{3}\leftrightarrow u=\frac{\pi}{3}$ from the usual 30-60-90 degree triangle. Get

$$\int_1^{\sqrt{3}}\frac{1}{x^2\,\sqrt{4-x^2}}\,dx=\int_{\pi/6}^{\pi/3}\frac{2\cos(u)\,du}{4\sin^2(u)\,2\cos(u)}=\int_{\pi/6}^{\pi/3}\frac{du}{4\sin^2(u)}=\frac{1}{4}\int_{\pi/6}^{\pi/3}\csc^2(u)\,du=$$
$$\frac{-1}{4}\cot(u)\Big|_{\pi/6}^{\pi/3}=\frac{1}{4}\left[\cot\left(\frac{\pi}{6}\right)-\cot\left(\frac{\pi}{3}\right)\right]=\frac{1}{4}\left[\sqrt{3}-\frac{1}{\sqrt{3}}\right]=\frac{1}{4}\frac{2}{\sqrt{3}}=\frac{1}{2\sqrt{3}}=\frac{\sqrt{3}}{6}.$$

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(2) (6 Points) Since $(x^2-x-2)=(x-2)(x+1)$ and (x^2+x+2) is irreducible ($b^2-4ac=1-8<0$), the denominator factors into irreducibles as $(x-2)^2(x+1)^2(x^2+x+2)$, so the correct form for the partial fraction is:

$$\text{c)}\ \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{Ex+F}{x^2+x+2}$$

Math 222 Calculus 2 Fall 2008 Exam 2 Solutions

(3) (8 Points) Does $\displaystyle\int_0^\infty \frac{2x}{1+x^4}\,dx$ converge or diverge? Why? Evaluate it if it converges.

This integral is improper since the upper bound is infinity. We find the indefinite integral $\displaystyle\int \frac{2x}{1+x^4}\,dx = \tan^{-1}(x^2) + C$ so the improper integral is the converging limit

$$\lim_{t\to\infty} \tan^{-1}(t^2) - \tan^{-1}(0^2) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

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(4) (8 Points) Does $\displaystyle\int_1^4 \frac{1}{x-2}\,dx$ converge or diverge? Why? Evaluate it if it converges.

This integral is improper since the denominator of the integrand is zero at $x = 2$. We must break it up into $\displaystyle\int_1^2 \frac{1}{x-2}\,dx + \int_2^4 \frac{1}{x-2}\,dx$. If either integral diverges, then so does the original. The indefinite integral is $\ln|x-2| + C$. Both integrals diverge. The first improper integral diverges since it is defined to be

$$\lim_{t\to 2^-} \int_1^t \frac{1}{x-2}\,dx = \lim_{t\to 2^-} (\ln|t-2| - \ln|1-2|) = -\infty.$$

The second improper integral diverges since it is defined to be

$$\lim_{s\to 2^+} \int_s^4 \frac{1}{x-2}\,dx = \lim_{s\to 2^+} (\ln|4-2| - \ln|s-2|) = \infty.$$

It is enough to show that one of these improper integrals diverges.

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(5) (8 Points) Use the Comparison Theorem to determine whether the following improper integral converges or diverges. DO NOT COMPUTE THE EXACT VALUE OF THE INTEGRAL, but show all work needed for the Comparison Theorem.

$$\int_2^\infty \frac{x}{\sqrt{x^6+4}}\,dx$$

For $2 \le x$ we have $x^6 < x^6 + 4$ so $0 < \dfrac{x}{\sqrt{x^6+4}} < \dfrac{x}{\sqrt{x^6}} = \dfrac{1}{x^2}$. We know that $\displaystyle\int_2^\infty \frac{1}{x^p}dx$ converges for $p > 1$, so $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$ converges. By the Comparison Theorem, the given integral converges.

Math 222 Calculus 2 Fall 2008 Exam 2 Solutions

(6) (25 Points, 5 Points Each) Determine whether each sequence $\{a_n\}_{n=1}^{\infty}$ converges or diverges, and if it converges, then find its limit.

(a) $a_n = \ln(2n + \sqrt{n}) - \ln(n) = \ln\left(\frac{2n+\sqrt{n}}{n}\right) = \ln\left(2 + \frac{1}{\sqrt{n}}\right) \to \ln(2)$ as $n \to \infty$ converges since $\frac{1}{\sqrt{n}} \to 0$.

(b) $a_n = \frac{2n^5 - n^3 + 3n}{3n^5 + 2n^2 - 3} = \frac{2 - \frac{1}{n^2} + \frac{3}{n^4}}{3 + \frac{2}{n^3} - \frac{3}{n^5}} \to \frac{2}{3}$ converges as $n \to \infty$.

(c) $a_n = \frac{\sqrt{n}}{\ln(n)} = f(n)$ where $f(x) = \frac{\sqrt{x}}{\ln(x)}$. Using L'Hospital's Rule ($\frac{\infty}{\infty}$-type):

$\lim_{x\to\infty} \frac{\sqrt{x}}{\ln(x)} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{x^{-1}} = \lim_{x\to\infty} \frac{\sqrt{x}}{2} \to \infty$ diverges.

(d) $a_n = \frac{\tan^{-1}(n+\sqrt{n})}{\sqrt{4+e^{-n}}} \to \frac{\pi/2}{\sqrt{4}} = \frac{\pi}{4}$ converges since $n + \sqrt{n} \to \infty$ and $e^{-n} \to 0$ as $n \to \infty$, and $\tan^{-1}(x) \to \frac{\pi}{2}$ as $x \to \infty$.

(e) $a_n = \sqrt[n]{3^n + 1}$ (Hint: Use $3^n + 1 < 3^n + 3^n$, $\lim_{n\to\infty} \sqrt[n]{2} = 1$, and Squeeze Th.)

Since $3^n < 3^n + 1 < 3^n + 3^n = 2(3^n)$ we get

$$3 = \sqrt[n]{3^n} < \sqrt[n]{3^n+1} < \sqrt[n]{3^n+3^n} = \sqrt[n]{2(3^n)} = (\sqrt[n]{2})(3).$$

This means that $3 \le a_n \le (\sqrt[n]{2})(3)$, and since $\lim_{n\to\infty} \sqrt[n]{2} = 1$, the Squeeze Theorem says the sequence converges to 3.

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(7) (5 Points) A **convergent** sequence of **positive** numbers satisfies the recursive relation $a_n a_{n+1} = a_n + 2$. Find $\lim_{n\to\infty} a_n$.

Taking the limit as $n \to \infty$ of the recursive relation, and using that $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} a_{n+1}$, get

$$(L)(L) = L + 2 \quad \text{so} \quad 0 = L^2 - L - 2 = (L-2)(L+1)$$

whose only positive solution is $L = 2$.